Looking at quantum mechanics with model theoretic glasses

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Quantum mechanics crash course

The world is probabilistic: nature can be in a state where the outcome of a measurement is not determined before the measurement happens.

A physical model is not a simulation, it is a "prediction machine".

The probabilities of possible outcomes are coded in a *state* or *wave function*. This is often presented as a unit vector in complex Hilbert space.

In particular, the position of a particle is described by a function ψ of a position variable $\bar{x} \in \mathbb{R}^n$ such that $|\psi(\bar{x})|^2$ encodes probabilities:

$$\int_{\mathbb{R}^n} |\psi(\bar{x})|^2 d\bar{x} = 1 \quad \text{and} \quad \mathbb{P}(\bar{x} \in E) = \int_E |\psi(\bar{x})|^2 d\bar{x}.$$

Dirac's view

Dirac: *Principles of Quantum Mechanics* (eds 1–4, 1930–1967)

- *states* as vectors (or the 1-dimensional space they span)
- *observables* as linear operators, whose eigenvalues correspond to the possible measurement outcomes
- the δ-function

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \qquad \delta(x) = 0, \text{ for } x \neq 0$$

  as a *notational simplification*
- from the 3rd edition onward, presentation in form of *bra*- and *ket*-vectors $\langle x|$, $|x\rangle$, where $|x\rangle$ is an eigenvector with eigenvalue x, and $\langle x|$ is its *dual vector* (a functional)
- Dirac explicitly points out, that he is not working in a Hilbert space

  > *The space of bra or ket vectors when the vectors are restricted to be of finite length and to have finite scalar products is called by mathematicians a Hilbert space. The bra and ket vectors that we now use form a more general space than a Hilbert space.*

(At least) two differing models of quantum physics

In beginning physics textbooks[1] the theory is presented via finite dimensional vector spaces

- self-adjoint operators have a basis of eigenvectors (*ket*-vectors)
- the space of study, called a *physical Hilbert space* is a space spanned by Dirac's δ-functions, where calculation rules from finite dimensional spaces apply

In mathematical physics textbooks the theory is presented via the spectral theorem of (unbounded) self-adjoint operators.

[1] e.g. R. Shankar *Principles of Quantum Mechanics 2nd ed*, Plenum Press, New York, 1994.

Using (a weak form of) Łos's theorem

Idea:

- Here we consider an infinite dimensional Hilbert space (with some operators) to be the correct model of physics, and want to approximate it by finite dimensional spaces (with operators).
- If we can embed the correct model into an ultraproduct of finite dimensional spaces, what is true in filter-many of them, will be true in the product.
- Even if the embedding is *not elementary*, we can still use it to justify *equations* in the image of the correct model.

A first approach: finding eigenvectors

Intended model

$$(L_2(\mathbb{R}), Q, P)$$

where

$$Qf(x) = xf(x) \quad \text{and} \quad Pf(x) = -i\hbar(df/dx)(x).$$

Note that the commutator is

$$[Q, P] = i\hbar$$

and this cannot happen in any finite dimensional Hilbert space.

Tools I

Theorem (Stone)

If $(U_t)_{t\in\mathbb{R}}$ is a continuous unitary representation of $(\mathbb{R},+)$, i.e. for all $t \in \mathbb{R}$,

- *U^t is a unitary operator in a complex Hilbert space H,*
- *$t \mapsto U^t(x)$ is continuous for all $x \in H$, and*
- *$U^{t+t'} = U^t U^{t'}$,*

then there is a unique self adjoint Q such that $U^t = e^{itQ}$ for all $t \in \mathbb{R}$.

This allows us to study unitary operators instead.

Tools II

Theorem (Stone–von Neumann)

The class of Hilbert spaces H with continuous unitary representations

$$t \mapsto U^t \quad \text{and} \quad t \mapsto V^t$$

satisfying the Weyl commutator law

$$V^w U^t = e^{i\hbar tw} U^t V^w$$

is essentially categorical in every cardinality.

This allows us to aim for the Weyl commutator law (for many but not all t, w) in the ultraproduct.

Tools III

Unbounded operators via ultraproducts

- standard metric ultraproducts are limited to operators with a common modulus of uniform continuity
- by splitting the space up into *nice enough* pieces, one can take ultraproducts without a uniform bound, and get partially defined (unbounded) operators in the ultraproduct model

The first approach in a nutshell

- Look at N-dimensional spaces with operators (H_N, P_N, Q_N) where
  - we have continuous unitary representations corresponding to P_N and Q_N,
  - the commutator relation holds only partially.
- Build a metric ultraproduct with partially defined operators such that
  - the commutator relation holds for a dense set of ts and ws
  - continuity of the representation holds only in a small submodel.
- Embed $(L_2(\mathbb{R}), Q, P)$ into the well-behaved part of the ultraproduct.
- Notice that the ultraproduct has eigenvectors built as ultraproducts of eigenvectors from the H_N spaces.

Pros and cons

What works
One can calculate with eigenvectors in the ultraproduct by looking at the what happens in *filter-many* finite-dimensional models (cf. physics approach of letting $N \to \infty$)

What doesn't
The answers one calculates are wrong.

The problem stems from there being continuum many eigenvectors for each position, stemming from different approximating paths. The calculations depend on $\sqrt{N}x$ being an integer (for N the dimension, and x a rational point), but although for every x, there are filter many Ns for which this happens, every model has infinitely many approximations of x that go wrong.

There is a remedy by calculating averages, but this is not the physics approach.

A second approach: distributions and Rigged Hilbert spaces

A rigged Hilbert space consists of a Hilbert space H and a subspace Φ of "test functions", with a finer norm on Φ.

The linear functionals over Φ are called *distributions*, and they form the *dual* of Φ, denoted Φ^*.

As $\Phi \subset H$, we have $H^* \subset \Phi^*$, and with $H^* = H$, one gets

$$\Phi \subset H \subset \Phi^*$$

For suitably chosen Φ, the functionals in Φ^* act as "generalised eigenvectors".

(Φ, H, Φ^*) is also called a *Gelfand triple*.

Example

Let $H = L_2(R)$, and let Φ the set of *Schwartz functions*, i.e., infinitely differentiable functions $\varphi : \mathbb{R} \to \mathbb{C}$ whose derivatives tend to 0 at infinity faster than any power of $\frac{1}{|x|}$.
Then for every $x \in \mathbb{R}$ the functional $f_x\varphi = \varphi(x)$ acts as a Dirac delta function corresponding to the value x.

Looking for distributions

In this approach we construct distributions within an ultraproduct of finite dimensional spaces.

The results so far only consider bounded self-adjoint operators, but we believe the method can be generalised to unbounded operators.

The spectral theorem, background

Definition

Let T be a bounded linear operator. The *spectrum* of T, denoted $\sigma(T)$ is the set of all $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ does not have a bounded inverse.

Fact: self-adjoint operators have real spectra.

Definition

Let H be a separable Hilbert space, and A a bounded self-adjoint operator. A vector φ is called *cyclic* for A if the vectors $A^n\varphi$, $n < \omega$, span a dense subset of H.

The spectral theorem

Theorem

If A is a bounded self-adjoint operator on H with a cyclic vector, then there exists a measure μ on $\sigma(A)$ and a unitary operator $U : H \to L_2(\mathbb{R}, d\mu)$ such that

$$UAU^{-1}\varphi(x) = x\varphi(x).$$

Note: in the non-cyclic case we get an orthogonal sum of such L_2 spaces.

Tools

Theorem (Stone)

There is a one-one correspondence between unitary operators U and self-adjoint operators A with spectrum $\subseteq [0,1]$ and not having 0 in the point spectrum, given by $U = e^{2\pi i A}$.

Fact

We can modify the above, to consider spectra $\subset [-\frac{\pi}{2}, \frac{\pi}{2}]$, and a correspondence $U = e^{iA}$.

So, we start with a bounded self-adjoint operator A, with a cyclic vector φ (of norm 1).

We assume $\sigma(A) \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$, and consider $U = e^{iA}$.

Now φ is cyclic also for U, in the sense that the vectors $U^k\varphi$, $k \in \mathbb{Z}$, span a dense set of H.

Finite-dimensional approximations of (H, U) with cyclic vector φ

$$\dots \; U^{-(N+1)}\varphi \quad U^{-N}\varphi \quad \dots \; U^{-1}\varphi \quad \varphi \quad U\varphi \quad \dots \; U^{N}\varphi \quad U^{N+1}\varphi \quad \dots$$

$$\dots \; U^{-(N+1)}\varphi \;\Big|\; U^{-N}\varphi \quad \dots \; U^{-1}\varphi \quad \varphi \quad U\varphi \quad \dots \; U^{N}\varphi \;\Big|\; U^{N+1}\varphi \quad \dots$$

U_N

Eigenvectors

In each H_N

- U_N is unitary and has an eigenvector basis $(u_N(k))_{k<2N+1}$ with corresponding eigenvalues $\lambda_N(k)$,
- the cyclic vector can be written as

$$\varphi = \sum_{k=0}^{2N} \xi_N(k) u_N(k)$$

where each $\xi_N(k)$ is a non-negative real, and $\sum_{k=0}^{2N} \xi_N(k)^2 = 1$.

Note: The spaces H_N extend each other, but the bases do not.

Spectral measure in H_N

Remember: in each H_N, $\varphi = \sum_{k=0}^{2N} \xi_N(k) u_N(k)$

Definition

For each $N < \omega$, define a measure μ_N for subsets $X \subset \mathbb{C}$:

$$\mu_N(X) = \sum_{k<2N, \lambda_N(k) \in X} \xi_N(k)^2$$

Note that for all $X \subset \mathbb{C}$, $\mu_N(X) \leq 1$, as $\|\varphi\| = 1$.

Spectral measure from ultraproduct

Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω.

We construct a spectral measure for U in steps:

1. For each $X \subseteq \mathbb{C}$, let $\mu^n(X)$ be the ultralimit $\lim_{\mathcal{U}} \mu_N(X)$
2. consider a set of *nice* vertical (I_r) and horizontal (J_r) lines for which for all $\delta > 0$ there is $\varepsilon > 0$ such that their "ε-thickenings" I_r^ε and J_R^ε have small "measure"

$$\mu^n(I_r^\varepsilon) < \delta, \quad \mu^n(J_r^\varepsilon) < \delta$$

3. define an outer measure based on the μ^n-value of boxes bounded by nice lines

$$\mu^*(Y) = \inf \left\{ \sum_{k=0}^{\infty} \mu^n(X_k) \mid X_k \text{ a } \textit{nice box}, Y \subseteq \bigcup_{k<\omega} X_k \right\}$$

4. by Caratheodory's construction, find a σ-algebra of sets for which μ^* is a measure

Two embeddings

One can embed

- on the one hand, (H, U) into the ultraproduct $\prod_{N<\omega}(H_N, U_N)/\mathcal{U}$ via the diagonal embedding of generating terms $U^z\varphi$ (these make sense $\mathcal{U}$-often)
- on the other, $L_2(S, \mu^*)$ into the ultraproduct via mappings of polynomials into the finite dimensional models

$$F_N(f_P) = \sum_{n<2N+1} \xi_N((n)f_P(\lambda_N(n))u_N(n)$$

Combining these give an isomorphism between (H, U) and $(L_2(S, \mu^*), U_{spectr})$.

Dissecting the ultraproduct construction

A metric ultraproduct is built in several steps:

1. form the product
2. throw out the infinite elements
3. find the subspace of infinitesimal elements and quotient them out
   this last part can be split in two: mod out the zeros, then the (other) infinitesimals

By changing the norm in steps 2–3, one can build spaces resembling the Φ and Φ^* of a Gelfand triple.

Pros and cons

What works

One can build distributions for *countably many positions* at a time, and (for nice enough operators) calculate with generalised eigenvectors as in beginning physics books. The construction builds into it an averaging technique that solves the problem from the eigenvectors of the first approach.

Drawbacks

The construction is rather messy, and doesn't exactly give a Gelfand triple.

Comparison and question

Our main motivation came from trying to understand Zilber's work on quantum mechanics.

Over the years it turned out, Zilber is not mimicking the L_2 model but is working in a pseudofinite vector space.

The big question is,

Is the correct model for physics finite, pseudofinite or continuous?